# *Measurement of Water in Clouds.*

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Since the initial data on which a weather prediction is to be based must, to be adequate, include the water-content of the clouds; and since observations of this are scanty, it will be well to survey what can be done in this respect. Three types of cloud can be measured: I. Clouds into which an observer can enter. Several observers, notably Conrad and independently Wagner, have measured the water in clouds on mountains by drawing a measured volume of atmosphere over absorbing substances. The information is summarised in Hann's 'Meteorology,' 3rd edition, p. 306. II. Cloud through which the sun's outline can be seen and which also exhibit coronæ, as they often do. III. Uniform stratus, provided that some way can be found for measuring the size of the particles.

## II. *Thin Clouds.*

For clouds through which the sun can be seen shining, it is remarkable that the edge of the sun's disc, if observed through dark spectacles, almost invariably appears quite sharp, free from any blurring.

I have measured the diminution by clouds of the difference of brightness at the edge of the sun's disc by a special photometer, arranged as follows and drawn diagrammatically in fig. 1. An image of the sun is formed on a piece of stout white paper by means of a lens fitted with an iris diaphragm. The paper is illuminated from behind by an electric lamp, except for a small area which is rendered opaque and is brought into coincidence with the sun's image. Thus the lamp produces a constant difference of brightness, opposite in sign to that produced by the sun's image. By varying the iris diaphragm, or by the introduction of absorbing screens, the difference of brightness, due to the sun, is made to balance that due to the lamp. The instrument is standardised on the unclouded sun at the same zenith distance. To eliminate difficulties due to difference of colour between lamp and sun, the paper was observed through a piece of copper-ruby glass. The instrument could be made convenient and portable. Stray light scarcely disturbed the balance, provided the enclosure containing the lamp was black, except for its plane white translucent front.

For the interpretation of the readings we may reason as follows:—The difference of brightness between the sun and the adjoining sky must be due to the light which comes straight through the cloud, and is independent of the general glow of scattered light which is superposed on the disc and its surroundings alike. The fraction of the incident light which can come straight through must depend only on the size and number of the holes in the sieve, to which we may liken the cloud.

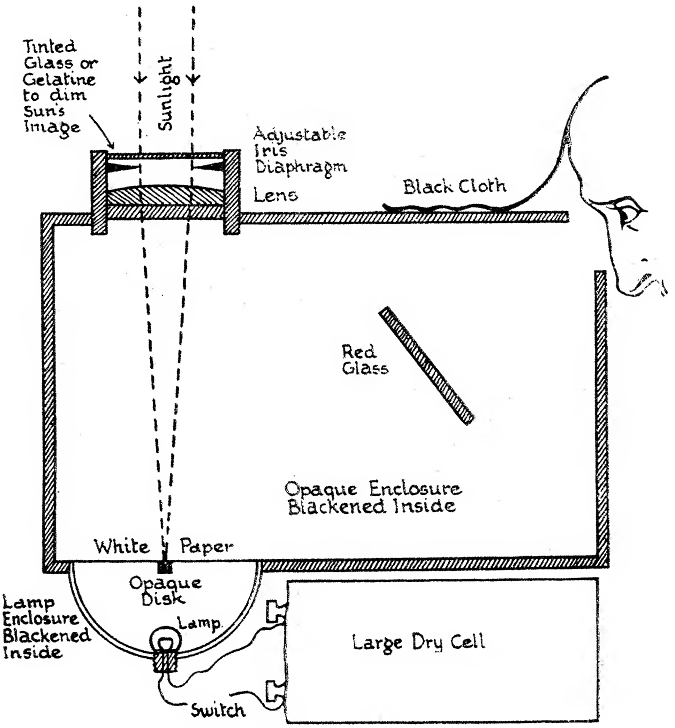
Tinted Glass or Gelatine to dim Sun's Image
Sunlight
Adjustable Iris Diaphragm
Lens
Black Cloth
Red Glass
Opaque Enclosure Blackened Inside
White Paper
Opaque Disk
Lamp Enclosure Blackened Inside
Lamp
Large Dry Cell
Switch

FIG. 1.—Diagram of Contrast Photometer.

Thus, consider a portion of the beam of light 1 cm.$^2$ in cross-section and of a length, $dl$, measured along the ray. Let there be a number, $n$, of particles per cubic centimetre, and let the particles be all of one size, and let A be the area of the diametral plane of a particle supposed spherical. Then the particles in $dl$ remove a fraction $Andl$ of the incident energy from the direct beam, by reflexion, refraction, or absorption. Diffraction may modify this result in the case of very small particles, but the ordinary cloud particle has a diameter some ten times longer than the wave-length of light, and for the present, diffraction has been neglected.

So if I be the difference of brightness at the sun's edge in the *direct* beam at any point

$$dI = -IAndl. \qquad (1)$$

If A$n$ be constant, we have on integrating

$$\log(\mathrm{I}/\mathrm{I}_0) = -\mathrm{A}nl, \tag{2}$$

where $l$ is the length of the ray passing through the cloud, and $\mathrm{I}_0$ is the initial difference of brightness.

Now, by the geometry of the sphere,

$$\mathrm{A}n = \tfrac{3}{2}\left\{\frac{\text{volume of particles per volume of atmosphere}}{\text{(diameter of particle)}}\right\}. \tag{3}$$

The product of the right side into $l$ should therefore remain constant if $\mathrm{I}/\mathrm{I}_0$ remains fixed. Now $\mathrm{I}/\mathrm{I}_0$ is some sort of measure of the mental sensation of contrast of the object with its surroundings, which will reach a small, more or less fixed, value when $l$ approaches the "distance of visibility" of a terrestrial object seen through a mist. Thus the above equation appears to be in harmony with the statement of W. Trabert[*] that the distance of visibility is proportional to the diameter of the particle, for a fixed water-content presumably. The equation also harmonises with Aitken's observation[†] that the product of the distance of visibility into the number of dust particles per volume, is nearly constant at a fixed humidity, but increases if the particles grow in size by the condensation of vapour upon them. Our equation finds further support in the observations of Conrad and Wagner,[‡] on the amount of water in certain clouds, when the data are compared with the diameters of the cloud particles measured by the same observers.[§] For the mean of their observations may be summarised in the following formula:—

$$\frac{\text{(Distance of visibility)} \times \text{(volume of water per volume of cloud)}}{\text{(Diameter of cloud particle)}} = 2{\cdot}9. \tag{3a}$$

If we may insert this number 2·9 into formulæ (3) and (2), it would mean that Conrad and Wagner considered an object just invisible when $\mathrm{I}/\mathrm{I}_0$, the difference of brightness between it and its surroundings, fell to $e^{-\frac{3}{2}\times 2\cdot 9} = \frac{1}{77}$ of its value in absence of mist. This seems a not improbable ratio for a terrestrial object.

Now if the cloud is a horizontal layer it will be convenient to put $dl = \sec\zeta\,.\,dh$ where $dh$ is the change of height corresponding to $dl$, and $\zeta$ is the sun's zenith distance.

With this substitution equation (1) becomes

$$d\mathrm{I}/\mathrm{I} = -\mathrm{A}n\sec\zeta\,.\,dh. \tag{4}$$

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[*] Hann's 'Meteorology,' 3rd edit., p. 260.

[†] Hann's 'Meteorology,' p. 16.

[‡] Hann, p. 306.

[§] Hann, p. 306.

On substituting the value of $An$ from (3) and integrating across the vertical thickness of the cloud, (4) yields

$$\log \frac{\mathrm{I}}{\mathrm{I}_0} = -\tfrac{3}{2} \sec \zeta \int \frac{(\text{volume of particles per volume of air})\, dh}{(\text{diameter of particle})}. \tag{5}$$

Now if the particles are all of one size, their diameter may be taken outside the integral in (5), which then comes out, no matter in what way the density of the cloud may vary with height, giving

$$\log_e \frac{\mathrm{I}}{\mathrm{I}_0} = -\tfrac{3}{2} \sec \zeta \frac{(\text{total volume of particles per horizontal area of cloud})}{(\text{diameter of particle})}. \tag{6}$$

If the particles are of water, not ice, this is the same as

$$\log_e \frac{\mathrm{I}}{\mathrm{I}_0} = -\tfrac{3}{2} \sec \zeta \times (\text{equivalent rainfall measured in diameters of a particle}), \tag{7}$$

which puts the result in its simplest form. Here I is the contrast of brightness at the sun's edge when seen through the cloud, and $\mathrm{I}_0$ is the contrast for the unclouded sun.

If the cloud is a mixture of particles of different sizes, represented by their diametral areas $\mathrm{A}_1$, $\mathrm{A}_2$, $\mathrm{A}_3$, ..., and if $n_1$, $n_2$, $n_3$, ..., are the respective numbers of these particles per volume of atmosphere, then (1) becomes

$$d\mathrm{I} = -\mathrm{I}\,(\mathrm{A}_1 n_1 + \mathrm{A}_2 n_2 + \mathrm{A}_3 n_3 + \ldots)\, dl, \tag{8}$$

and the second member of (6) is made up of a corresponding series of terms, the total volume of each size of particles per horizontal area being divided by the diameter and the quotients being added.

It is conceivable that observations of coronæ, if sufficiently thorough, might enable us to disentangle the terms due to the different diameters. But in the present paper it is assumed that the droplets are all of one size.

My observational results are as follows:—

One noticed, by naked eye, a sensible dimming of the sunlight when $\mathrm{I}/\mathrm{I}_0 = 0{\cdot}7$, as it is for the thinnest cirrus. At the other end of the scale, the sun's disk was just discernible through stratus, when $\mathrm{I}/\mathrm{I}_0 = 1/10{,}000$.

Thus, for the clouds cirrus, cirro-stratus, and cirro-cumulus, the average value of the volume of particles per horizontal area appears to be of the order of the radius of the particle. The range of values for the diameter found by Pernter, from observations of coronæ, was $0{\cdot}5 \times 10^{-3}$ cm. to $2{\cdot}0 \times 10^{-3}$ cm.* Inserting these, together with the density of ice, we get for the mass of ice per horizontal square centimetre

$$0{\cdot}22 \times 10^{-3} \quad \text{to} \quad 0{\cdot}9 \times 10^{-3} \text{ grm.-cm.}^{-2}.$$

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* Hann's 'Meteorology,' p. 257.

| Description of cloud. | $-\frac{2}{3}\cos\zeta \,.\, \log_e (I/I_0) = \dfrac{\text{Volume of particles per horizontal area}}{\text{(diameter of particle)}}$. |
|---|---|
| Faintest cirrus | 0·07. |
| Very thin cirrus | 0·3, 0·3. |
| Ci or ci-stratus | 0·04. |
| Very thin ci-stratus | 0·06, 0·2, 0·8, 0·3, 0·5, 0·3, 0·8, 0·6, 0·4, 0·4, 0·8. |
| Ci-stratus, thin | |
| Ci-stratus (typical ?) | |
| Ci-cumulus | 0·6. |
| Ci-cumulus + ci-stratus | 0·8, 0·9, 2·1. |
| Alto-cumulus | 0·5. |
| Stratus, sun much dimmed, but still obvious at $\zeta = 40°$ | 2·5. |
| Stratus, sun's disk just visible at $\zeta = 49°$ | 4·1. |

As a caution, it must again be noted that diffraction ought to be considered before this result is relied upon.

## III. *Thick Stratus.*

For uniform stratus it is customary to make a very rough eye estimate of the water-content expressed in such words as "thin, heavy, dark, ugly, lowering." Can photometric measurements yield a corresponding numerical quantity ?

The apparatus which I have used for this purpose was made as follows: A wedge of blotting paper was fixed inside a blackened cigar box, where it could be observed through a small hole covered with red gelatine. The two sides of the wedge were illuminated by the light passing through two adjustable iris diaphragms. The diaphragms were covered by pieces of white paper. One piece of paper was illuminated by a small electric lamp, which served as an intermediate standard. The other piece of paper was illuminated either by the unclouded sun, which served as the absolute standard, or alternatively by a small definite solid angle of stratus cloud. The iris diaphragms were adjusted until the two sides of the wedge appeared equally bright. In this way the light from a small solid angle of sky was found. In calculating the light from the whole sky, there is a trap of which it is well to beware. The solid angle of a hemisphere is $2\pi$, but the light falling on a horizontal area from a uniformly bright sky is only $\pi$, not $2\pi$ times the light from unit solid angle, on account of obliquity. For if $\zeta$ is the zenith distance, the solid angle between $\zeta$ and $\zeta + d\zeta$ is $2\pi \sin\zeta \,.\, d\zeta$; and the projection of the horizontal area normal to the same direction is proportional to cos $\zeta$. So the light falling on the horizontal area is proportional to the brightness of the sky at $\zeta$ multiplied by $2\pi \sin\zeta \cos\zeta \, . \, d\zeta$, which is equal to $-\frac{1}{2}\pi d(\cos 2\zeta)$, the integral of which from $\zeta = 0$ to $\zeta = \frac{1}{2}\pi$ is $\pi$.

This also shows that a correct way to average the observations is to plot them as a function of cos $2\zeta$ and to find the area under the curve, for this procedure gives proper weight to the different zones. The light falling on an area of ground must be equal to the light leaving an equal area of cloud, if the cloud is uniformly bright and if the scattering of the intervening air be neglected. Again, neglecting scattering, the light falling on the upper surface of the cloud is proportional to the cosine of the sun's zenith distance, which can be found from the date and the time. [Scattering can be allowed for by reference to the Tables of L. V. King,* based on the Smithsonian observations.] In this way I found that a strato-nimbus cloud of ordinary appearance transmitted a quarter of the red light falling on its upper surface.

A preliminary interpretation of such a result may be developed from Schuster's theory of "Radiation through a Foggy Atmosphere."†

Following Schuster, let us divide directions into two large groups, in our case by a horizontal plane.

Let E be the sum of all radiation which has a downward component.

E′ the sum of all radiation which has an upward component.

Let *dh*, as usual, be an element of length drawn vertically upward. As a rough allowance for the variety of directions of the rays, we may put the path of the ray as $\surd 2dh$, corresponding to a slope of 45°. Then if C is the fraction of the radiation which has its vertical component reversed in sign by the scattering in unit length of ray, we have

$$\frac{dE'}{\surd 2 \, . \, dh} = C\,\{E - E'\}, \qquad (9)$$

$$\frac{dE}{\surd 2 \, . \, dh} = C\,\{E - E'\}, \qquad (10)$$

whence

$$\frac{d(E - E')}{dh} = 0 \qquad (11)$$

and

$$\frac{d}{\surd 2 \, . \, dh}(E + E') = 2\,C\,(E - E'). \qquad (12)$$

Now, at the lower surface of the cloud, the upward radiation bears to the downward radiation a ratio equal to the reflectivity of the earth, vegetation, or sea. Putting B for this reflectivity, and letting subscript $_i$ distinguish values at lower surface of the cloud, we have

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* 'Phil. Trans.,' A, 1913.

† 'Astrophys. Jour.,' vol. 21, No. 1, Jan., 1905.

$$BE_i = E_i'. \tag{13}$$

$$E_i + E_i' = E_i(1+B); \qquad E_i - E_i' = E_i(1-B). \tag{14, 15}$$

But, by (11), $E-E'$ is independent of height, so that in general

$$E-E' = E_i(1-B), \tag{16}$$

$$E+E' = 2E-(E-E') = 2E-E_i(1-B). \tag{17}$$

Consequently equation (12) becomes, since $E_i$ does not depend on $h$,

$$\frac{d}{\sqrt{2}\,dh}2E = 2C\,E_i(1-B). \tag{18}$$

Integrating upwards from the lower surface,

$$E = \sqrt{2}\,.\,E_i(1-B)\int_i C\,dh + \text{const.} \tag{19}$$

and the constant of integration must be

$$E_i. \tag{20}$$

So that at the top of the cloud, denoted by a subscript zero,

$$E_0 = \sqrt{2}\,.\,E_i\left\{1+(1-B)\int_i^0 C\,dh\right\}. \tag{21}$$

Again, at the top of the cloud the downward component of direct sunlight is

$$E_0 = S\cos\zeta, \tag{22}$$

where $\zeta$ is the sun's zenith distance and S the intensity for a zenith sun.

Therefore
$$\frac{S\cos\zeta}{E_i} = \sqrt{2}\left\{1+(1-B)\int_i^0 C\,dh\right\}. \tag{23}$$

Now $E_i/S\cos\zeta$ is the fraction of the incident light which is transmitted by the stratus cloud, and is a quantity which can be observed.

Another quantity required for equation (23) is B, the reflectivity of the earth's surface. This has been observed by a simple photometer consisting of a white wedge in a black box, the sides of the wedge being illuminated by lights from diametrically opposite directions, admitted to the box through adjustable iris diaphragms, one opening to the earth, the other to uniform stratus. In this way I found, for directions near the zenith and nadir, the following ratios of brightness of ground, to brightness of stratus, for red light. Fresh snow, 0·65; dry chalk in Eastern Champagne, France, 0·2; sea-sand, damp but not glistening, Seaview, Isle of Wight, 0·13; grass, 0·15, 0·06; sea, grey green, between Portsmouth and the Isle of Wight, $0{\cdot}06_3$; stubble after hay harvest, 0·04; green oats on chalk, 0·03; loam, damp dark brown, 0·03; dark green bushes, $0{\cdot}01_5$. Except in the cases of snow and water these numbers may be taken as first approximations to B. For snow and water the reflectivity depends notably on direction, and the instrument which has just been described does not enable us to deal effectively with that variation, because it contains no reference standard.

Of the quantities in equation (23) it remains to consider $\int_i^0 \mathrm{C}\,dh$ where C is the fraction of the radiation which has its vertical component reversed in sign by the scattering in unit length of ray, and where the integration is taken across the vertical thickness of the cloud.

To find C we need in the first place the fraction of direct sunlight turned back by a single drop of water. As previously we have only considered two groups of directions called "upwards" and "downwards" for short, so here we only need a division into "forwards" and "backwards," the division being made by a plane through the drop at right angles to the incident sunlight. However, it has not been found possible to obtain a convincing answer to the question: what fraction of the light is turned backwards? without exploring the distribution of light in detail. This is not to be regretted, for the detail will be required in any future thorough treatment of the problem, in which directions are divided into infinitesimal groups instead of into two hemispheres.

It will be convenient to express the intensity of the scattered radiation by a quantity which depends only on the angle between the incident and the scattered radiation, and which has unity for its mean value. Such a quantity is $\psi$ defined by

$$\psi = \frac{\text{(candle-power of drop in any direction)}}{\text{(mean spherical candle-power of drop)}}. \qquad (24)$$

Now $\psi$ is independent of the unit chosen for candle-power, and therefore we may, if we like, interpret candle-power to mean $r^2$ times the energy per second falling normally on unit area placed at a great distance, $r$, from the drop. Then, since the area of a sphere of this radius is $4\pi r^2$, it follows that the mean spherical candle-power is equal to the whole energy falling on this large sphere, divided by $4\pi$. And since the drop is transparent, the total energy scattered on to the large sphere is equal to the total energy striking the drop. Thus we reach a new expression for $\psi$, more suited to calculation

$$\psi = \frac{\left\{\begin{matrix}\text{energy per second scattered on to unit area normal} \\ \text{to ray at a distance } r\end{matrix}\right\} \times 4\pi r^2.}{\text{(total energy per second on drop)}} \qquad (25)$$

The distribution of $\psi$ is shown in figs. (2) and (3). The polar diagram, fig. (2), represents a solid of revolution rather like a prolate spheroid with

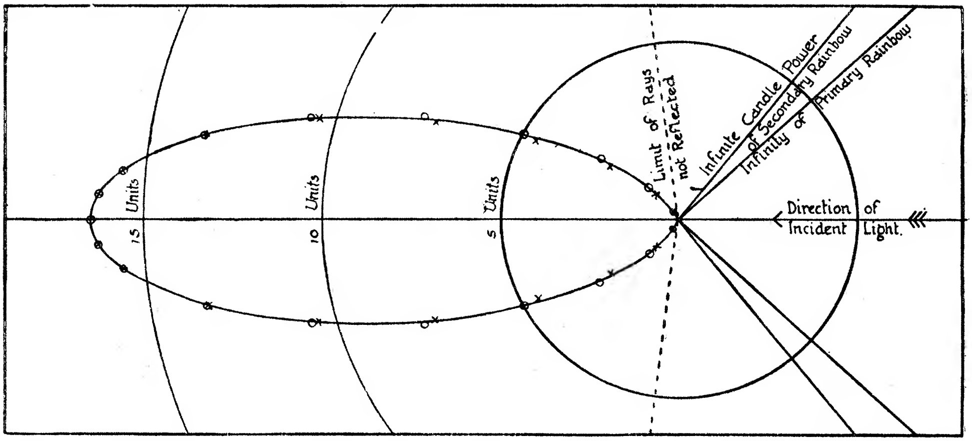

FIG. 2.—Polar Diagram of Candle-power of a Sphere of Water Illuminated by Parallel Light. Refractive index 1·332. Diffraction neglected.

The unit is the mean-spherical-candle power. The line represents the distribution for unpolarized light ; the crosses for light polarized so that the magnetic vector is in the plane of incidence, the circles when the electric vector is in the plane of incidence.

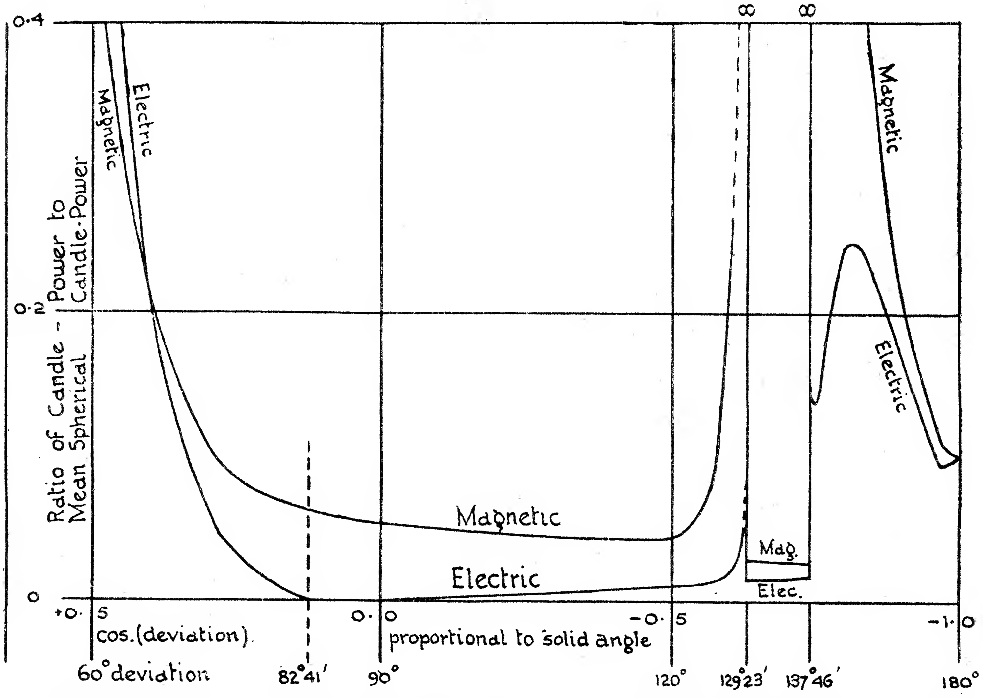

FIG. 3.—Candle Power of a Sphere of Water Exposed to Parallel Incident Light : Detail for Deviations 60° to 180°.

The polarization is indicated by stating the vector, magnetic or electric, which lies in the plane of incidence. Refractive index 1·332. Diffraction is neglected.

two conical shells attached to one end. It shows that the scattered light does indeed mainly go forward or backward rather than sideways, thus so far justifying the division of directions into two hemispheres. The backwardly directed light, which is too small to show on this scale, except at the infinities, is exhibited in fig. (3), in which the abscissa is proportional to area on the sphere at infinity.

If there were a uniform stratus cloud so thin that its droplets did not hide behind each other, then its brightness ought to be proportional to $\psi$, multiplied by the secant of the zenith distance of the portion of cloud observed. In this connection it is interesting to note that the polarisation at 90° from the sun, for a single drop, is such that $\psi = 0{\cdot}0543$ when the magnetic vector lies in the plane containing the sun, the cloud, and the observer; and by $\psi = 0{\cdot}0028$ when the electric vector is in this plane. Here $\psi$ is proportional to the intensity of the light, and there is of course no relation between the phases of these polarisations, so that the light is not quite plane-polarised.

Now, as to the method by which these diagrams and numbers have been computed. A ray of light striking a sphere of water at an angle of incidence, $\alpha$, is divided into two, one part reflected, the other entering the drop at an angle of refraction here called $\beta$.

The entering ray is further subdivided by repeated impacts on the surface, at each of which the angle between the ray and the normal to the surface is the same, being $\alpha$ for the external ray, $\beta$ for the internal. Now it is known* that if incident light is polarised in the plane of incidence it remains so after reflection or refraction by water. The same is true for light polarised at right angles to the plane of incidence, but not for polarisations in intermediate planes. The reflectivity depends upon the polarisation, being, according to Fresnel's formulæ

$$\left[\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}\right]^2 \quad \text{or} \quad \left[\frac{\tan(\alpha-\beta)}{\tan(\alpha+\beta)}\right]^2 \qquad (25, 26)$$

respectively, according as the magnetic or the electric vector is in the plane of incidence. If then either of these vectors lie in the diametral plane of the sphere at the point where the ray first strikes it, the reflectivity will remain the same at each successive impact, and the polarisation of the emergent portions of the ray will have the same relation to the diametral plane in which the ray lies. For this reason these two polarisations must be computed separately. The desired result for unpolarised light is obtained at the end of the process by taking the mean.

To simplify the treatment of the infinities which represent the rainbows, it is convenient to imagine the sun replaced by an ideal star, infinitely small and bright.

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* Drude's 'Optics,' Chap. II.

The angle of incidence, $\alpha$, may be used as a measure of position on the water-sphere. The portion of the sphere included in $d\alpha$ catches a fraction, $\sin 2\alpha \,.\, d\alpha$, of the whole energy which strikes the sphere. This energy is distributed among various rays, deviated from the incident direction through angles $\gamma$. The area of the portion lying within $d\gamma$ of a very large sphere of radius $r$ is $2\pi r^2 \sin\gamma \,.\, d\gamma$. Thus, if all the light falling on $d\alpha$ went to this single portion $d\gamma$, then $\psi$ would be equal to

$$\frac{2\sin 2\alpha \,.\, d\alpha}{\sin\gamma \,.\, d\gamma}. \tag{27}$$

Actually $\psi$ is the sum of a number of such terms, one for every fraction of the ray, and each multiplied by $\sigma$ for every reflection, and by $(1-\sigma)$ for every refraction which this portion of the ray has undergone, where $\sigma$ is the reflectivity corresponding to $\alpha$ and to the polarisation.

The deviations, $\gamma$, of the successive fractions are easily shown to be

$$\left.\begin{array}{ll} \text{Once reflected} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \gamma_1 = 2\alpha - \pi, \\ \text{Refracted, refracted} \ldots\ldots\ldots\ldots\ldots\ldots & \gamma_2 = 2(\alpha - \beta). \\ \text{Refracted, reflected, refracted} \ldots\ldots\ldots & \gamma_3 = 2(\alpha - \beta) + \pi - 2\beta, \\ \text{Refracted, twice reflected, refracted} \ldots & \gamma_4 = 2(\alpha - \beta) + 2(\pi - 2\beta). \end{array}\right\} \tag{28}$$

The light more than twice internally reflected has been neglected. This is justifiable, because its intensity would be multiplied by $\sigma^3(1-\sigma)^2$, or by smaller quantities, none of which become appreciable except over a zone which catches very little incident light.

Now, when we attempt to count up areas under the curves in fig. 3, in order to find the radiation going backwards, we are baffled by the presence of the infinities. But the difficulty about infinite candle-powers may be entirely avoided by replacing integrations over the sphere-at-infinity by corresponding integrations over the surface of the drop. Thus, the light turned "backwards" is made up of—

(i) Light once reflected at angles of incidence between 0° and 45°. This, expressed as a fraction of the whole light striking the drop, amounts to

$$\int_{\alpha=0}^{\alpha=45^\circ} \sigma \,.\, d(\sin\alpha)^2, \tag{29}$$

because $(\sin\alpha)^2$ is proportional to the whole amount of light striking the drop for angles of incidence between 0 and $\alpha$. The reflectivity, $\sigma$, computed from formulæ (25) and (26), runs as follows, for a refractive index of 1·332:—

| Angle of incidence. | Magnetic vector in plane of incidence. | Electric vector in plane of incidence. |
|---|---|---|
| 0° | 0 ·0203 | 0 ·0203 |
| 5° | 0 ·0205 | 0 ·0200 |
| 10° | 0 ·0212 | 0 ·0193 |
| 20° | 0 ·0244 | 0 ·0165 |
| 30° | 0 ·0308 | 0 ·0119 |
| 40° | 0 ·0429 | 9 ·0058 |
| 45° | 0 ·0528 | 0 ·0028 |
| 50° | 0 ·0665 | 0 ·0005 |
| 53° 6 ·2′ | 0 ·0779 | 0 ·0000 |
| 60° | 0 ·1147 | 0 ·0043 |
| 70° | 0 ·2191 | 0 ·0473 |
| 80° | 0 ·4564 | 0 ·2388 |
| 85° | 0 ·6733 | 0 ·4932 |
| 90° | 1 ·0000 | 1 ·0000 |

On plotting these as a function of $(\sin \alpha)^2$, or what is equivalent and more convenient, as a function of $\cos 2\alpha$, since $d(\sin \alpha)^2 = -\frac{1}{2} d(\cos 2\alpha)$, it is found that the integral (29) comes to 0·0111 for the mean of the two polarisations. That is to say, for unpolarised light; for the reflectivity is not the ratio of the vectors after and before reflection, but the ratio of their squares.

(ii) The primary rainbow and other light once reflected inside the drop. All this goes "backwards." The intensity is reduced in the ratio $\sigma$ at the reflection, and in the ratio $(1-\sigma)$ at each of the two refractions. So this part, as a fraction of the whole light incident on the drop, amounts to

$$\int_{\alpha=0}^{\alpha=90^\circ} \sigma (1-\sigma)^2 \, d(\sin \alpha)^2. \tag{30}$$

(iii) The secondary rainbow and most of the other light, which is twice reflected inside the drop, but not quite all of it, for one of the equations (28) shows that $\gamma_4$ is less than 90° when $\alpha$ lies between 0° and 39°. Accordingly, this backwardly directed part, as a fraction of the whole light incident on the drop, amounts to

$$\int_{\alpha=39^\circ}^{\alpha=90^\circ} \sigma^2 (1-\sigma)^2 \, d(\sin \alpha)^2. \tag{31}$$

(iv) The tertiary rainbow is directed forward. The higher rainbows are negligible.

Now adding integrals (30) and (31), we get

$$\int_{\alpha=0}^{\alpha=90^\circ} \sigma (1+\sigma)(1-\sigma)^2 \, d(\sin \alpha)^2 - \int_{\alpha=0}^{\alpha=39^\circ} \sigma^2 (1-\sigma)^2 \, d(\sin \alpha)^2. \tag{32}$$

The first integral in (32) has been evaluated as an area on a diagram for the two polarisations separately, with the result that the mean for unpolarised light is 0·0491. The second integral in (32) appears as a small correction, which reduces the total from 0·0491 to 0·0488.

Adding to 0·0488 the integral (29) in amount 0·0111, it follows that a fraction

$$0{\cdot}0599$$

of the sunlight striking the drop is scattered into the hemisphere which we call the backward direction.

We are now in a position to evaluate $\int C\,dh$, which is required for equation (28). Here C is the fraction of the radiation turned backward by the scattering in unit length of ray. But the whole radiation incident upon drops in a length $dl$ of ray is equal to the intensity of radiation multiplied by $And l$, where A is the area of the diametral plane of a drop, and $n$ the number of drops per volume. Therefore

$$C = 0{\cdot}0599\,An.$$

Then $\int C\,dh$ taken across a cloud $= 0{\cdot}0599\int An\,dh$, which, as before, is equal to

$$\tfrac{3}{2} \times 0{\cdot}0599 \times \frac{\text{(volume of liquid per horizontal area of cloud)}}{\text{(diameter of particle)}}, \tag{33}$$

and from (23) this is equal to

$$\left\{\frac{S\cos\zeta}{\sqrt{2E_i}} - 1\right\}\frac{1}{1-B}, \tag{34}$$

which gives us what we require.

For instance, for a strato-nimbus from which rain fell on the afternoon of 24 May, 1918, I found by photometer measurements with red light of wave-length 0·6 to 0·8 microns, that $S\cos\zeta/E_i$ was equal to 4·0, the sun's elevation being such that $\cos\zeta$ was 0·75. The reflectivity of the grass land was of the order of 0·15. Inserting these numbers in the formulæ, we find that the *volume of liquid per horizontal cm.*$^2$ *of cloud amounted to* 24 *diameters of the cloud-droplets.*

This figure is greater, as it should be, than the largest number of diameters, namely, four, determined by the "contrast" method for a stratus which just allowed the sun's disk to be seen.

Possible developments are the improvement of the above treatment by taking a less crude, but still finite, division of directions in space; or else by proceeding to infinite subdivision by means of Fredholm's integral equation,* having a kernel involving the function $\psi$, which is shown in figs. 2 and 3. A method for measuring the diameter of the cloud-droplets is also needed.

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* 'Cambridge Tracts in Mathematics,' No. 10, by Maxime Bôcher.